NIRMA UNIVARSITY

INSTITUTE OF TECHNOLOGY

DEPARTMENT OF MATHEMATICS & HUMANITIES

B. TECH. I, SEMESTER I (ALL BRANCHES)

MA101, CALCULUS

HANDOUTS

Module 1: Differential Calculus

Limits:

Consider A_n as the area of the inscribed polygon with 'n' sides. As 'n' increase, it is observed that A_n approaches to the area of the circle. The area of the circle obtained is the limit of the areas of the inscribed polygons and we denote it by

$$\lim_{n \to \infty} A_n = A.$$

Definition: The function $f : A \to R$, where $A \subset R$ is said to have limit L at $x = a$ (a may or may not belong to A) if given $\varepsilon > 0$, there exists $\delta > 0$, depending upon a and ε such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Notation: $\lim_{x \to a} f(x) = L$.

Properties: If $\lim_{x \to p} f(x)$, $\lim_{x \to p} g(x)$ (for the last property it must be nonzer0)exists. Then

$$
\begin{aligned}
\lim_{x\to p} (f(x)+g(x)) &= \lim_{x\to p} f(x) + \lim_{x\to p} g(x) \\
\lim_{x\to p} (f(x)-g(x)) &= \lim_{x\to p} f(x) - \lim_{x\to p} g(x) \\
\lim_{x\to p} (f(x)\cdot g(x)) &= \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x) \\
\lim_{x\to p} (f(x)/g(x)) &= \lim_{x\to p} f(x) / \lim_{x\to p} g(x)
\end{aligned}
$$

<u>Application</u>: The concept of a limit is necessary in order to understand the workings of the differential as well as the integral calculus. Limits are used in differentiation while finding the approximation for the slope of a line at a particular point, as well as in integration while finding the area under a curve. Also it is very useful to solve the problems of tangent, velocity, acceleration and the problems of engineering.

Limit is also used in deciding the nature of an infinite series which is helpful in Fourier series, Fourier integral and Z-transform, which gives Electrical Engineers an idea of signals of communications.

<u>Differentiation</u>

The derivative of the function *f* at *a* is the <u>limit</u>

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

where $f'(a)$ is a notation of a derivative.

<u>BASIC FORMULAE</u>

1 $\frac{d}{dx} x^n = n\, x^{n-1}$

2 $\frac{d}{dx} e^x = e^x$

3 $\frac{d}{dx} a^x = a^x \log a$

4 $\frac{d}{dx} \log x = \frac{1}{x}$,

5 $\frac{d}{dx} \log_a x = \frac{1}{x \log a}$

6 $\frac{d}{dx} \sin x = \cos x$

7 $\frac{d}{dx} c = 0$

8 $\frac{d}{dx} \tan x = \sec^2 x$

9 $\frac{d}{dx}\cos x = -\sin x$

10 $\frac{d}{dx}\sec x = \sec x \tan x$

11 $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

12 $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

13 $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x . \cot x$

14 $\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

15 $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

16 $\frac{d}{dx}(c)\, f(x) = c.\frac{d}{dx} f(x)$

17 $\frac{d}{dx}\{c_1 f_1(x) \pm c_2 f_2(x)\} = c_1 \frac{d}{dx} f_1(x) \pm c_2 \frac{d}{dx} f_2(x)$

18 $\frac{d}{dx}\{f_1(x).f_2(x)\} = f_1'(x) f_2(x) + f_1(x) f_2'(x)$

19 $\frac{d}{dx}\left\{\frac{f_1(x)}{f_2(x)}\right\} = \frac{f_1'(x) f_2(x) - f_1(x) f_2'(x)}{\{f_2(x)\}^2}$

<u>Successive Differentiation</u>

If we have y = f(x), then the notations used for the successive derivatives of y with respect to x are

y', y'', y''', ..., $y^{(n)}$,...

- Some standard formulae of nth derivative:

Function	**n^th derivative**
$y = e^{ax}$	$y^{(n)} = a^n e^{ax}$
$y = b^{ax}$	$y^{(n)} = a^n b^{ax} (\log b)^n$
$y = (ax+b)^m$ if m = -1	$y^{(n)} = m(m-1)...(m-n+1) a^n (ax+b)^{m-n}$ $y^{(n)} = \frac{(-1)^n n! a^n}{(ax+b)^n}$

$y=\log(ax+b)$	$y^{(n)}=\dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$
$y=\sin(ax+b)$	$y^{(n)}=a^n\sin\left(\dfrac{n\pi}{2}+ax+b\right)$
$y=\cos(ax+b)$	$y^{(n)}=a^n\cos\left(\dfrac{n\pi}{2}+ax+b\right)$
$y=e^{ax}\sin(bx+c)$ $y=e^{ax}\cos(bx+c)$	$y^{(n)}=\left(a^2+b^2\right)^{n/2}e^{ax}\sin\left(bx+c+n\tan^{-1}\left(\dfrac{b}{a}\right)\right)$ $y^{(n)}=\left(a^2+b^2\right)^{n/2}e^{ax}\cos\left(bx+c+n\tan^{-1}\left(\dfrac{b}{a}\right)\right)$

Taylor and Maclaurin's Series Expansion:

Taylor's series can be written in several forms.

The first is: $f(x)=f(a)+\dfrac{(x-a)}{1!}f'(a)+\dfrac{(x-a)^2}{2!}f''(a)+\dots$

The Incremental Form assumes one is evaluating the function *f(h)* at *f(h+x)*.

$$f(h+x)=f(h)+xf'(h)+\frac{x^2}{2!}f''(h)+\dots.$$

The Maclaurin series is a special case of the Taylor series where *h*, above, is zero:

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\dots$$

Some standard Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n},\ for\,|x| < 1$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n},\ for\,|x| < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,\ for\ all\ x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,\ for\ all\ x$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,\ for\ all\ x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,\ for\ all\ x$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots,\ for\,|x| < \frac{\pi}{2}$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots,\ for\ all\,|x| < 1$$

$$\tan^{-1} x = \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \dots,\ for\ all\,|x| < 1, x \neq \pm i$$

$$\binom{\alpha}{n} = \prod_{k=1}^{n} \frac{\alpha - k + 1}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,\ for\,|x| < 1$$

Partial differentiation:

Function of two variables:

A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by f(x,y) .The set D is the domain of f and its range is the set of values that f takes on , that is , $\{f(x,y)|(x,y) \in D\}$.

Function of n variables:

A function of n variables is a rule that assigns a number $z = f\left(x_1, x_2, x_3, \dots, x_n\right)$ to an n- tuples.

Limit of function of two variables:

Let f be a function of two variables *x* and *y*, whose domain D includes points close to *(a, b)*. Then we say that the limit of *f(x, y)* approaches *(a, b)* is L and we write $\lim_{(x,y)\to(0,0)} f(x,y)=L$ if for every number $\varepsilon>0$ there is a corresponding number $\delta>0$ such that if $(x,y)\in D$ and $0<\sqrt{(x-a)^2+(x-b)^2}<\delta$ then $|f(x,y)-L|<\varepsilon$.

Continuity:

A function of two variables is said to be continuous a (a, b) if ,

$$\lim_{(x,y)\to(0,0)} f(x,y)=f(a,b) .$$

We say f is continuous on D if f is continuous at every point (a, b) in D.

Partial Derivative:

If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Notations for partial derivatives: If $z=f(x,y)$, we write

$$f_x(x,y)=f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial z}{\partial x}=f_1=D_1f=D_xf$$

$$f_y(x,y)=f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}f(x,y)=\frac{\partial z}{\partial y}=f_2=D_2f=D_yf$$

Rules for finding Partial Derivatives:

1. To find f_x, regard y as a constant and differentiable *f (x, y)* with respect to x.

2. To find f_y , regard x as a constant and differentiable $f(x, y)$ with respect to y.

Higher Ordered Derivatives

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Equation of Tangent Planes:

An equation to the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Equation of normal line to the surface through:

$$\frac{x - x_0}{\left(\frac{\partial f}{\partial x}\right)_P} = \frac{y - y_0}{\left(\frac{\partial f}{\partial y}\right)_P} = \frac{z - z_0}{\left(\frac{\partial f}{\partial z}\right)_P}$$

Differential (Total derivative):

$$dz = f_x(x, y)dx + f_y(x, y)dy$$
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Chain Rule (CASE 1): Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Chain Rule (CASE 2): Suppose that $z = f(x, y)$ is a differentiable function of *x* and *y*, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of *s* and *t*. Then z is a differentiable function of *s* and *t*. Then

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Maximum and Minimum Values:

A function of two variables has a local maximum at *(a, b)* if $f(x, y) \le f(a, b)$ when *(x, y)* is near *(a, b)*. The number *f (a, b)* is called a local maximum value. If $f(x, y) \ge f(a, b)$ when *(x, y)* is near to *(a, b)*, then *f (a, b)* is called a local minimum value.

Second Derivative Test: Suppose the second partial derivatives of *f* are continuous on a disk with center *(a, b)*, and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ Let

$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2$

- **(a)** If D>0 and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum.
- **(b)** If D>0 and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum.
- **(c)** If D<0 and $f(a, b)$ is neither maximum nor minimum.

Method of Lagrange Multipliers:

To find the maximum and minimum values of $f(x, y, z)$ subject to the constrain $g(x, y, z) = k$

[Assuming that these extreme values exist and $\nabla g \ne 0$ on the surface $g(x, y, z) = k$]

(a) Find all values of *x*, *y* ,*z* and λ such that $\nabla f(x\ y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$

(b) Evaluate f at all the points (x, y, z) that result from step (a) .The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Euler's Theorem:

If u is a function of x and y of degree n, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$.

Cor. 1 If u is a homogenous function of x *and* y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Cor. 2 If Z is a homogenous function of degree n in x and y and Z = $f(u)$ then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Cor. 3 If Z is a homogenous function of degree n in x and y and Z = $f(u)$ then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)(g'(u)-1) \text{ } where\, g(u)=n\frac{f(u)}{f'(u)}$$

Derivative of an implicit function:

If $f(x, y) = 0$ be an implicit function with $y = g(x)$, then $\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{f_x}{f_y}$

Taylor's Expansion for a function of two variables:

$$f(x+h,y+k)=f(x,y)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f+\ldots$$

Cor.1

$$f(a+h,b+k)=f(a,b)+\left[hf_x(a,b)+kf_y(a,b)\right]+\frac{1}{2!}\left[h^2 f_{xx}+2hkf_{xy}(a,b)+k^2 f_{yy}(a,b)\right]+\ldots$$

Cor.2

$$f(a+h,b+k)=$$
$$f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+$$
$$\frac{1}{2!}\left[(x-a)^2 f_{xx}+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]+\ldots$$

Jacobian:

1. If $u = u\,(x, y)$ and $v = v\,(x, y\,)$ then the Jacobian of u and v w.r.t x and y is given by $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

2. If $u = u\,(x, y\,,z)$ and $v = v\,(x, y, z)$ and $w = w(x, y, z)$, then the Jacobian of u and v and w w.r.t x and y and z is given by $J = \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$

Properties of Jacobian:

1. If u and v are functions of x and y and x and y are functions of r and s , then $\dfrac{\partial(u,v)}{\partial(r,s)} = \dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,s)}$

2. If $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ and $J' = \dfrac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$.

Module 2: Integral Calculus

Proper integral:

An integral which has neither limit infinite and from which the integrand does not approach infinity at any point in the range of integration.

Improper Integral:

An integral is an improper integral if either the interval of integration is not finite (improper integral of type 1) or if the function to integrate is not continuous (not bounded) in the interval of integration (improper integral of type 2).

Example:-1. $\int_0^\infty e^{-x}\,dx$. is an improper integral of type 1 since the upper limit of integration is infinite.

Example:-2. $\int_0^1 \frac{1}{x}\,dx$. is an improper integral of type 2 because $\frac{1}{x}$ is not continuous at 0.

Example:-3. $\int_0^\infty \frac{1}{x-1}\,dx$. is an improper integral of types 1 since the upper limit of integration is infinite. It is also an improper integral of type 2 because $\frac{1}{x-1}$ is not continuous at 1 and 1 is in the interval of integration.

Reduction formula:

(1) $$\int_0^{\frac{\pi}{2}} \sin^m x\,dx = \frac{m-1}{m}\,\frac{m-3}{m-2}\,\frac{m-5}{m-4}.......x \quad \text{where } x = \begin{cases} 1 & \text{if } m \text{ is odd} \\ \frac{\pi}{2} & \text{if } m \text{ is even} \end{cases}$$

(2) $$\int_0^{\frac{\pi}{2}} \cos^m x\,dx = \frac{m-1}{m}\,\frac{m-3}{m-2}\,\frac{m-5}{m-4}.......x \quad \text{where } x = \begin{cases} 1 & \text{if } m \text{ is odd} \\ \frac{\pi}{2} & \text{if } m \text{ is even} \end{cases}$$

(3) $\int_0^{\frac{\pi}{2}} \cos^m x \sin^n x\, dx = \frac{(m-1)(m-3)........(n-1)(n-3)(n-5)..........}{(m+n)(m+n-2)(m+n-4)............} x$

$$\text{where } x = \begin{cases} \frac{\pi}{2} & if\ m \,\&\, n\, are\ even \\ 1 & ottherwise \end{cases}$$

❖ Rectification:

A curve whose length can be found is called a rectifiable curve and the process of finding the length of a curve is called rectification.

(i) Length of the plane curve in Cartesian form: Let $y = f(x)$ be the equation of the plane curve. Let S be the length of the arc of the plane curve $y = f(x)$ included between two points A and B whose abscissa are a and b. Then S can be given as

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(ii) Length of the plane curve in Polar form: Let $r = f(\theta)$ be the polar equation of the curve. Then the length of the arc of the curve included between two points whose vectorial angles are $\theta = \alpha$ and $\theta = \beta$ is

$$S = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(iii) Length of the plane curve in parametric form: Let $x = f(t)$ and $y = g(t)$ be the equation of the curve in parametric form, where t is a parameter. then the length of the curve between the points $t = t_1$ and $t = t_2$ is

$$S = \int_{t=t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

❖ **Quadrature:** The process of finding the area of a bounded region of a curve is called quadrature.

(i) Area of a plane region in Cartesian form: Let $y=f(x)$ be a function defined on the interval $[a,b]$. Let us assume that $f(x)\geq 0$. Then the area A of the curvilinear Trapezoid bounded by the curve $y=f(x)$, the x- axis and the two ordinates $x=a$ and $x=b$ is given by

$$A=\int_{x=a}^{b} f(x)\,dx$$

The area A bounded by the two curves $y=f_1(x)$, $y=f_2(x)$ and the ordinates $x=a$ and $x=b$ is given by

$$\begin{aligned} A &= \int_{x=a}^{b} f_2(x)\,dx-\int_{x=a}^{b} f_1(x)\,dx \\ &= \int_{x=a}^{b}\left[f_2(x)-f_1(x)\right]dx \end{aligned} \quad \text{, provided } f_2(x)\geq f_1(x)$$

To find the area of a closed curve we have to find out the tangents to the curve parallel to the y-axis. Let $x=a$ and $x=b$ be two tangents. Let an intermediate ordinate meet the curve in two points $P_1(x,y)$ and $P_2(x,y)$ where $y_1>y_2$, then the area A of the closed curve is

$$A=\int_{x=a}^{b}(y_1-y_2)\,dx$$

Where the values of y_1 and y_2 corresponding to any value of x are found by solving the equation of the curve as a quadratic in y.

(ii) Area bounded by a Polar Curve: The area bounded by the curve $r=f(\theta)$ between the radii vector $\theta=\alpha$ and $\theta=\beta\,(\alpha\leq\beta)$ is given by

$$A=\frac{1}{2}\int_{\theta=\alpha}^{\beta} r^2 d\theta = \frac{1}{2}\int_{\theta=\alpha}^{\beta}\left[f(\theta)\right]^2 d\theta$$

(iii) <u>Area bounded by a Parametric Curve:</u> If $x=f(t)$ and $y=g(t)$, $a\le t\le b$, are the parametric equations of a curve, then the area bounded by the curve, the x-axis and the ordinates $x=a$ and $x=b$ is given by

$$A=\int_{x=a}^{b} f(x)dx=\int_{x=a}^{b} g(t)\frac{d}{dt}\left[f(t)\right]dt$$

❖ **<u>Volume of the solid of revolution:</u>** A solid of revolution is generated when we revolve a plane region R about a line L. The line L is called the axis of revolution. For example, a plane region R bounded by $y=f(x)$, *x*-axis, $x=a$ and $x=b$ is rotated about x-axis then we get a solid. Such a solid is called solid of revolution.

(a) The volume of the solid generated by the revolution about the *x*- axis of the area bounded by the curve $y=f(x)$, the ordinates $x=a$, $x=b$ and the *x*-axis is given by

$$V=\pi\int_a^b y^2 dx.$$

(b) The volume of the solid generated by revolution about the *y*-axis of the area bounded by the curve $x=g(y)$, the abscissa $y=c$, $y=d$ and the *x*-axis is given by

$$V=\pi\int_c^d x^2 dy.$$

<u>Beta-Gamma function:</u>

<u>Gamma function:</u>

The gamma function (also known as Euler's integral of the second kind) is denoted by Γn and defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx; \quad n > 0$$

<u>Properties of Gamma function:</u>

(1) $\Gamma(n+1) = n\Gamma n \quad ; n > 0$.

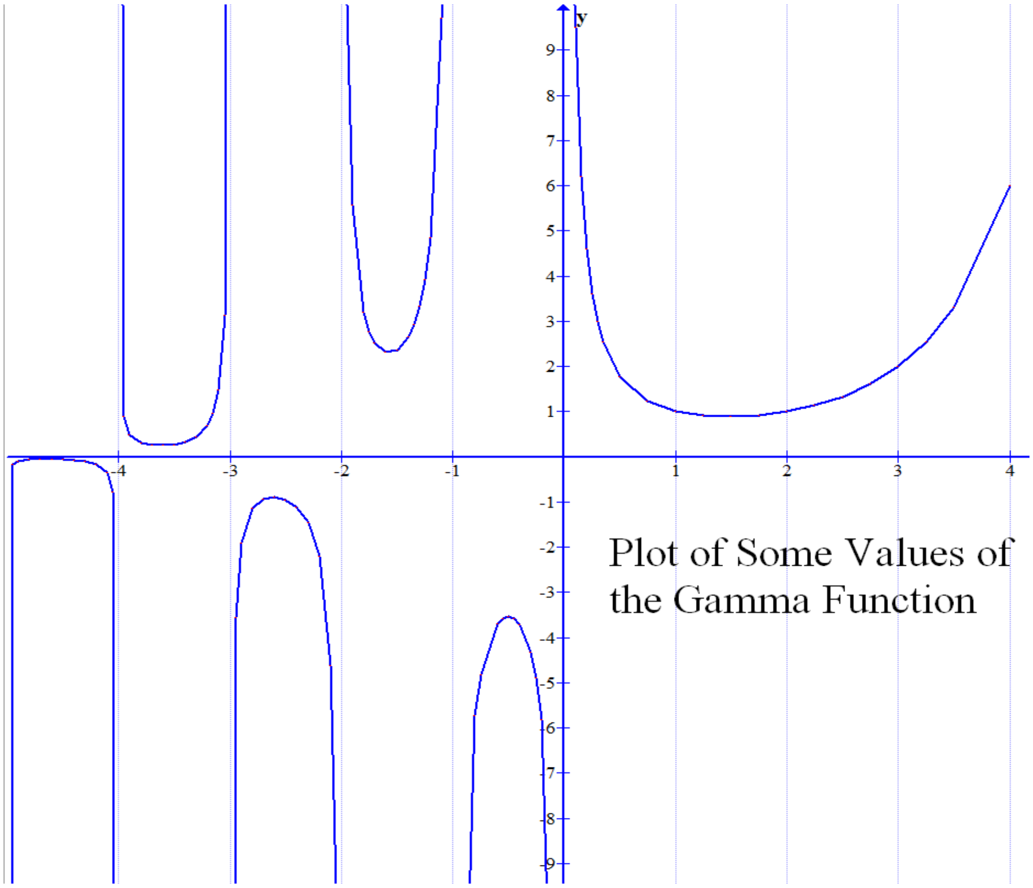
(2) $\Gamma(n+1) = n!$, when n is positive integer.

(3) $\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx \quad ; n > 0$.

(4) $\dfrac{\Gamma n}{a^n} = \int_0^\infty e^{-ax} x^{n-1} dx \; ; a > 0, n > 0.$

(5) $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

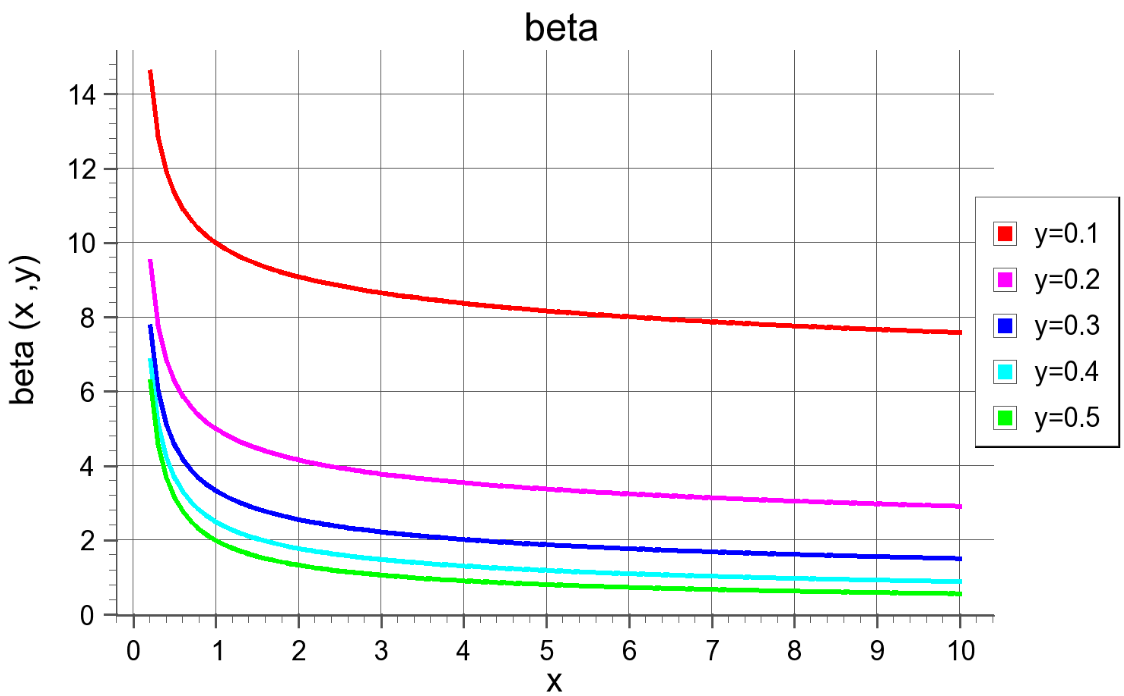
(6) $\int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$



Beta function:

The beta function (also known as Euler's integral of first kind) is denoted by $B(m,n)$ or $\beta(m,n)$ and defined as

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx\ ;m>0,n>0\,.$$



Properties of Beta function:

(1) $B(m,n) = B(n,m)$

(2) $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

(3) $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

(4) $B(m,n) = \int_0^{1} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Relation between Beta and Gamma functions:

$$B(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma (m+n)}$$

Some standard Results:

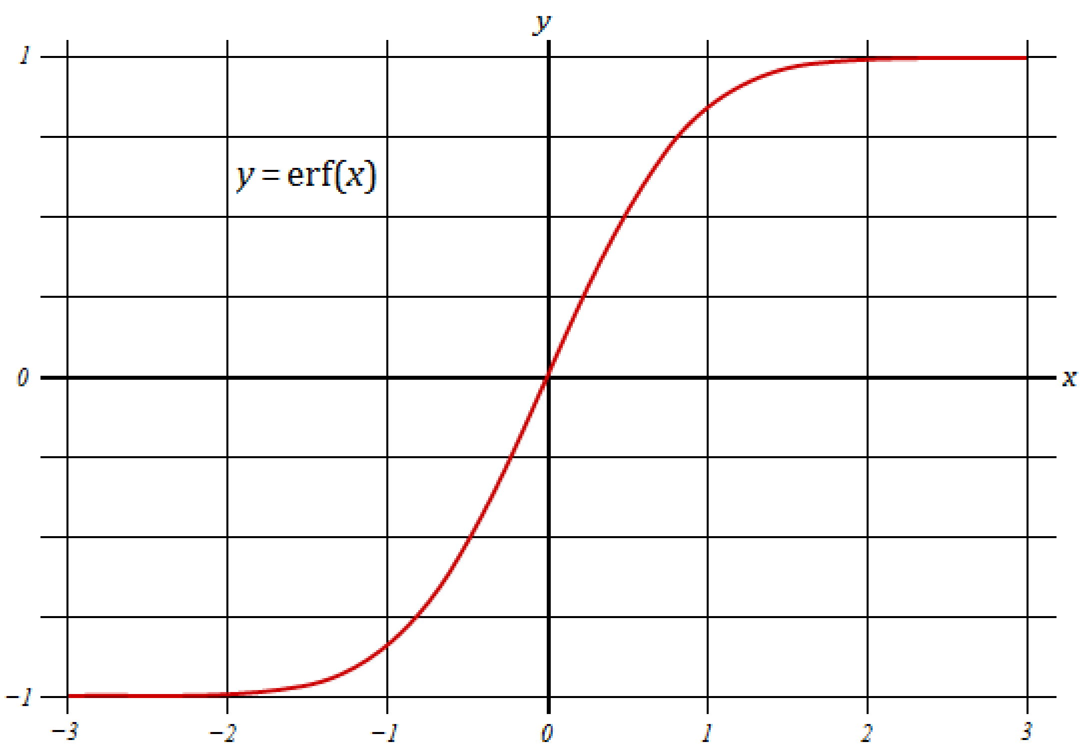
(1) $\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$.

(2) $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\ ; 0 < n < 1$ (Euler's formula).

(3) $\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$ (Legendre's formula or Duplication formula)

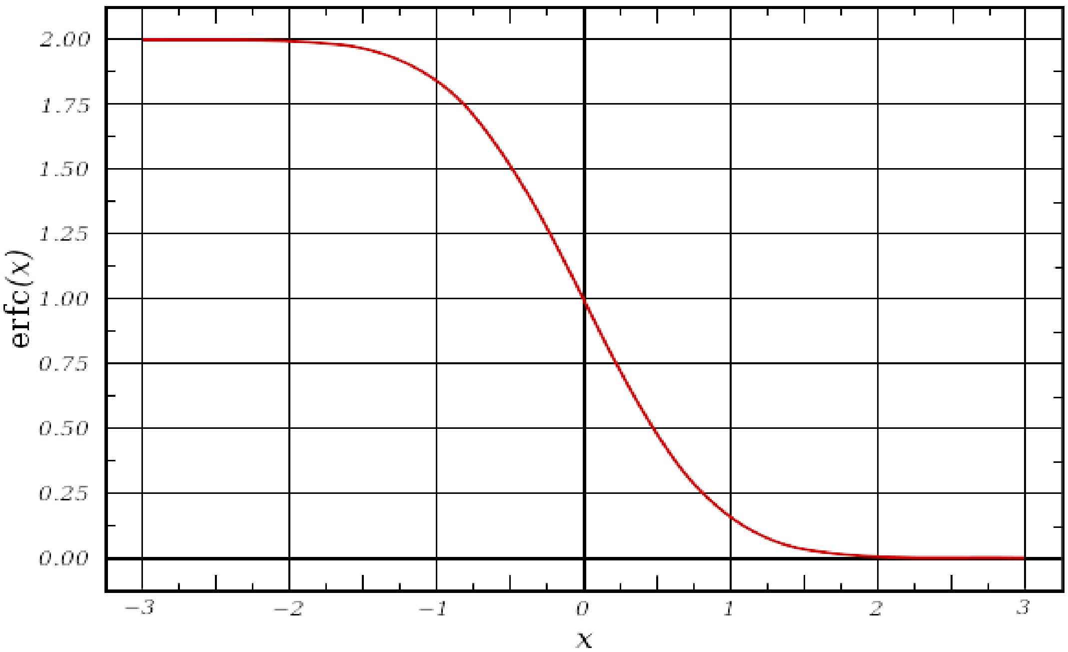
Error function:

The error function of x is defined by $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$



Complimentary error function:

The complimentary error function of x is defined by $erf_c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$



Properties:

(1) $erf(0) = 0$, $erf_c(0) = 1$.

(2) $erf(\infty) = 1$, $erf_c(\infty) = 0$.

(3) $erf(x) + erf_c(x) = 1$.

(4) $erf(x)$ is an odd function.

(5) $erf_c(x) + erf_c(-x) = 2$.

(6) $erf(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{1!\,3} + \frac{x^5}{2!\,5} - \frac{x^7}{3!\,7} + \ldots..\right)$

<u>Multiple integral:</u>

the double integral indicates that the infinitesimal summation is being carried out over a two-dimensional surface (in x and y directions).

Incidentally, if the wall does not have a convectional (rectangular) shape then its area can be calculated similarly according to

$$\text{Area} = \iint_{wall} dxdy$$

Another illustration is provided by quantum mechanics where the modulus squared of the wave function, $|\psi(x,y,z)|^2$, of an electron (say) gives the probability density of finding it at some point in space. The chances that the electron is in a small (cuboids) region of volume $\delta x \delta y \delta z$ is then $|\psi(x,y,z)|^2 \delta x \delta y \delta z$. Hence, the probability of finding it within a finite domain V is given by

$$\text{Probability} = \iiint |\psi(x,y,z)|^2 \, dxdydz \,,$$

which is known as Triple integral or volume integral.

❖ Area by double integration:

a) Cartesian coordinates:

The area A of a region R in $XOY-$plane bounded by the curves $y = f_1(x), y = f_2(x)$ and the lines $x = a, x = b$ is

$$A = \iint_R dy\,dx = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$$

b) Polar coordinates

The area A of a region R bounded by the curves $r = f_1(\theta), r = f_2(\theta)$ and the radii vectors $\theta = \alpha, \theta = \beta$ is

$$A = \iint_R dx\,dy = \int_\alpha^\beta r\,dr\,d\theta$$

❖ Volume as a double integrals:

$V = \iint z\,dx\,dy,$ Where $z = f(x,y)$ in Cartesian form

$V = \iint z\, r\, dr\, d\theta,$ Where $z = f(r,\theta)$ in polar form

- **<u>Volume of solid of revolution:</u>**

$V = 2\pi \iint y\, dy\, dx,$ Where $y = f(x)$ is a plane curve in Cartesian form.

$V = 2\pi \iint r^2 \sin\theta\, d\theta\, dr,$ Where $r = f(\theta)$ is polar curve (when rotated about initial line)

$V = 2\pi \iint r^2 \cos\theta\, d\theta\, dr,$ Where $r = f(\theta)$ is polar curve (when rotated about the line $\theta = \frac{\pi}{2}$)

- **<u>Volume as a triple integral:</u>**

The volume V of a three dimensional region is $V = \iiint dx\, dy\, dz$