Nirma University

Institute of Technology

Department of Mathematics and Humanities

B. Tech. (ALL), Semester I

MA101-Calculus

Applications of Calculus

Prepared by: Prof. Hanuman Tivari, Prof. Sikha shah, Prof. Brajesh Jha, Prof. Payal Oza,

Prof. Bijal Yeolekar, Prof. Monika Gupta, Prof. Dhara Patel

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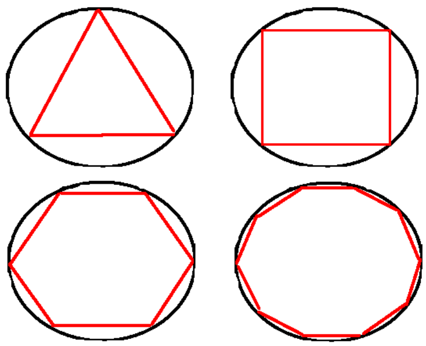
Differential Calculus

Limits: Calculus got introduced by the Greeks before around 2500 years, who found areas using the method of exhaustion. They used to find the area of any polygon by dividing it into triangles, but the same was little difficult for the curves. The Greeks "method of exhaustion" was to inscribe polygons in any figure, circumscribe polygons about the figure and then increasing the number of sides of the polygon.

Let us consider the case of a circle with inscribed regular polygons as shown in the figure:



Consider A_n as the area of the inscribed polygon with 'n' sides. As 'n' increase, it is observed that A_n approaches to the area of the circle. The area of the circle obtained is the limit of the areas of the inscribed polygons and we denote it by

$$\lim_{n\to\infty} A_n = A.$$

Definition: The function $f : A \to R$, where $A \subset R$ is said to have limit L at $x = a$ (a may or may not belong to A) if given $\varepsilon > 0$, there exists $\delta > 0$, depending upon a and ε such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Notation: $\lim_{x\to a} f(x) = L$.

Properties: If $\lim_{x\to p} f(x)$, $\lim_{x\to p} g(x)$ (for the last property it must be nonzer0)exists. Then

$$
\begin{aligned}
\lim_{x\to p}\left(f(x)+g(x)\right) &= \lim_{x\to p} f(x) + \lim_{x\to p} g(x)\\
\lim_{x\to p}\left(f(x)-g(x)\right) &= \lim_{x\to p} f(x) - \lim_{x\to p} g(x)\\
\lim_{x\to p}\left(f(x)\cdot g(x)\right) &= \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x)\\
\lim_{x\to p}\left(f(x)/g(x)\right) &= \lim_{x\to p} f(x) / \lim_{x\to p} g(x)
\end{aligned}
$$

Application: The concept of a limit is necessary in order to understand the workings of the differential as well as the integral calculus. Limits are used in differentiation while finding the approximation for the slope of a line at a particular point, as well as in integration while finding the area under a curve. Also it is very useful to solve the problems of tangent, velocity, acceleration and the problems of engineering.

Limit is also used in deciding the nature of an infinite series which is helpful in Fourier series, Fourier integral and Z-transform, which gives Electrical Engineers an idea of signals of communications.

Differentiation

The derivative of the function f at a is the limit

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

where $f'(a)$ is a notation of a derivative.

BASIC FORMULAE

1 $\frac{d}{dx} x^n = n\, x^{n-1}$

2 $\frac{d}{dx} e^x = e^x$

3 $\frac{d}{dx} a^x = a^x \log a$

4 $\frac{d}{dx} \log x = \frac{1}{x},$

5 $\frac{d}{dx} \log_a x = \frac{1}{x \log a}$

6 $\frac{d}{dx} \sin x = \cos x$

7 $\frac{d}{dx} c = 0$

8 $\frac{d}{dx} \tan x = \sec^2 x$

9 $\frac{d}{dx} \cos x = -\sin x$

10 $\frac{d}{dx} \sec x = \sec x \tan x$

11 $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

12 $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

13 $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x.\cot x$

14 $\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

15 $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

16 $\frac{d}{dx}(c)\, f(x) = c.\frac{d}{dx} f(x)$

17 $\frac{d}{dx}\{c_1 f_1(x) \pm c_2 f_2(x)\} = c_1 \frac{d}{dx} f_1(x) \pm c_2 \frac{d}{dx} f_2(x)$

18 $\frac{d}{dx}\{f_1(x).f_2(x)\} = f_1'(x) f_2(x) + f_1(x) f_2'(x)$

19 $\frac{d}{dx}\left\{\frac{f_1(x)}{f_2(x)}\right\} = \frac{f_1'(x) f_2(x) - f_1(x) f_2'(x)}{\{f_2(x)\}^2}$

Applications:

- We use the derivative to determine the maximum and minimum values of field variables (e.g. cost, strength, amount of material used in a building, profit, loss, etc.).
- In Chemical Engineering for finding the Work done under isothermal process using the differential equations as follows:

$$\frac{dW}{dV} = p;\ p \to pressure,\ W \to work\ done\ under\ isothermal,\ V \to Volume$$

- In Mechanical Engineering for finding the Force acted on a particle using the differential equations as follows:

$$F = \frac{dW}{dx};\ F \to Force,\ W \to Work\ done,\ x \to distance$$

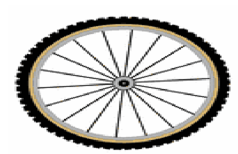
- In Electronics, Instrumentation Controls, Electrical Engineering for finding the voltage drop, current and charge using the differential equations:

$$I = \frac{dQ}{dt},;\ I \to current,\ Q \to charge,\ t \to time$$

$$V = \left(L\frac{dI}{dt}\right);\ V \to Voltage\ drop,\ L \to inductance$$

- ❖ If we are traveling in a car around a corner and we hit something slippery on the road (like oil, ice, water or loose gravel) and our car starts to skid, it will continue in a direction tangent to the curve.

- ❖ Likewise, if we hold a ball and swing it around in a circular motion then let go, it will fly off in a tangent to the circle of motion.
- ❖ The spokes of a wheel are placed normal to the circular shape of the wheel at each point where the spoke connects with the center.



Successive Differentiation

If we have y = f(x), then the notations used for the successive derivatives of y with respect to x are

y', y'', y''', ..., $y^{(n)}$,...

- ➢ Some standard formulae of nth derivative:

Function	**nth derivative**
$y=e^{ax}$	$y^{(n)}=a^n e^{ax}$
$y=b^{ax}$	$y^{(n)}=a^n b^{ax}(\log b)^n$
$y=(ax+b)^m$ if m = -1	$y^{(n)}=m(m-1)...(m-n+1)a^n(ax+b)^{m-n}$ $y^{(n)}=\dfrac{(-1)^n n!a^n}{(ax+b)^n}$
$y=\log(ax+b)$	$y^{(n)}=\dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

$y=\sin(ax+b)$	$y^{(n)}=a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$
$y=\cos(ax+b)$	$y^{(n)}=a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$
$y=e^{ax}\sin(bx+c)$ $y=e^{ax}\cos(bx+c)$	$y^{(n)}=\left(a^2+b^2\right)^{n/2}e^{ax}\sin\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right)$ $y^{(n)}=\left(a^2+b^2\right)^{n/2}e^{ax}\cos\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right)$

Applications:

- In Chemical Engineering : Successive differentiation is applied in amperometric titrations and some kinds of thermal analysis and kinetic experiments. Second Order derivative is used in ph-titrations
- In Instrumentation & Control: Trace analysis is based on fourth order derivative which is applied in order to take care to optimize signal-to-noise ratio.
- In Pharmacy: Fourth order derivative is used in the Pharmaceutical preparation of medicines in the Pharmaceutical industry.

Taylor and Maclaurin's Series Expansion: The concept of a Taylor series was formally introduced by the English mathematician Brook Taylor in 1715. If the Taylor series is centered at zero, then that series is also called a Maclaurin series, named after the Scottish mathematician Colin Maclaurin, who made extensive use of this special case of Taylor series in the 18th century.

In mathematics, a Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

Taylor's Series is based on the fact that, if a function is continuous and differentiable, the value of that function at small distance *h* from point *x* will be equal to the value of the function at *x*, plus a "fudge factor," or really a series of fudge factors. This is stuff you should know, because it is used extensively in math, physics, and geophysics.

Taylor's series can be written in several forms.

The first is: $f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$

The Incremental Form assumes one is evaluating the function *f(h)* at *f(h+x)*.

$$f(h+x) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \ldots.$$

Usually, in problems in applied physics/math, x is a small number, so only the first two terms are kept:

$$f(h+x) = f(h) + xf'(h)$$

This amounts to saying that the value of the function at $h+x$, namely, $f(h+x)$ will be equal to *f(h)*, plus a term that represents the slope of the function at *h*, times the distance *x*.

The Maclaurin series is a special case of the Taylor series where *h*, above, is zero:

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

Some standard Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \; for\,|x| < 1$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \; for\,|x| < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \; for\; all\; x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots, \; for\; all\; x$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \; for\; all\; x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots, \; for\; all\; x$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots, \; for\,|x| < \frac{\pi}{2}$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots, \; for\; all\,|x| < 1$$

$$\tan^{-1} x = \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots, \; for\; all\,|x| < 1, x \neq \pm i$$

$$\binom{\alpha}{n} = \prod_{k=1}^{n} \frac{\alpha - k + 1}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \; for\,|x| < 1$$

Applications:

- The effectiveness in error determination, function optimization, definite integral resolution, and limit determination is evidence of the Taylor series being an enormous tool in physical sciences and in Computational science as well as an effective way of representing complicated functions.
- A simple sensitivity analysis technique used in the science and engineering fields is presented for its application to memory less electric circuits. It was shown that a simple Taylor series expansion can be used in sensitivity.
- Approximating a numerical value of $f(t)$ by its series to approximate the value of the series for an analytic function $f(t)$ at some t within its interval of convergence, we can use a Taylor Polynomial of degree N.
- The Taylor series expansions (TSE) spreadsheets evaluate the statistical uncertainty in functions of up to 15 variables and evaluate equations consisting of random variables.TSE can accurately calculate the uncertainty for some non-linear functions .Someone with experience in using electronic spreadsheets should be able to use the provided spreadsheet layout, macros, cell functions, and range names to analyze the uncertainty for a user-supplied formula. Although the user-supplied formula is restricted to 15 random variables, a method of by passing this variable limit is furnished. While it was shown that the TSE spreadsheet does perform well, situations may arise where the Taylor series expansion analysis evaluated function is nonlinear or contains distributions which are highly skewed. Increasing the TSE analysis to a higher order (third or fourth order) may eliminate some of the discrepancies by Taylor series analysis.
- Taylor series is useful in Vibrations/Instrumentation System Dynamics.
- Taylor series is useful in solving the state space model as :
  Models of dynamic systems with concentrated parameters are commonly represented using sets of first-order ordinary differential equations (ODEs). We call these models State -space model, $x(t) = f(x(t), u(t), t)$ where x is the state vector, u is the input vector, and t denotes the time, the independent variable across which we wish to simulate. We also require initial conditions for the state variables: $x(t = t_0) = x_0$.

Partial differentiation:

In practical applications, most of the quantities depend on more than one variable. For example volume of a rectangle depends on length, breadth and height; temperature of a point on earth depends on longitude and latitude at that point etc. So we need to extend the idea of calculus of single variable to calculus of several variables. Although rules of calculus remain essentially the same, partial derivatives come into existence. We can find partial derivatives in the studies of probability, statistics, fluid dynamics, electricity etc. The list of some of the applications of partial derivatives is as follows:

Function of two variables:

A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by f(x,y) .The set D is the domain of f and its range is the set of values that f takes on , that is , $\{f(x,y)|(x,y)\in D\}$.

Function of n variables:

A function of n variables is a rule that assigns a number $z=f\left(x_1,x_2,x_3,\dots,x_n\right)$ to an n- tuples.

Limit of function of two variables:

Let f be a function of two variables *x* and *y*, whose domain D includes points close to *(a, b)*. Then we say that the limit of *f(x, y)* approaches *(a, b)* is L and we write $\lim\limits_{(x,y)\to(0,0)} f(x,y)=L$ if for every number $\varepsilon>0$ there is a corresponding number $\delta>0$ such that if $(x,y)\in D$ and $0<\sqrt{(x-a)^2+(x-b)^2}<\delta$ then $|f(x,y)-L|<\varepsilon$.

Continuity:

A function of two variables is said to be continuous a (a, b) if ,

$$\lim_{(x,y)\to(0,0)} f(x,y)=f(a,b)\ .$$

We say f is continuous on D if f is continuous at every point (a, b) in D.

Partial Derivative:

If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Notations for partial derivatives: If $z=f(x,y)$, we write

$$f_x(x,y)=f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial z}{\partial x}=f_1=D_1 f=D_x f$$

$$f_y(x,y)=f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}f(x,y)=\frac{\partial z}{\partial y}=f_2=D_2 f=D_y f$$

Rules for finding Partial Derivatives:

1. To find f_x, regard y as a constant and differentiable *f (x, y)* with respect to x.

2. To find f_y , regard x as a constant and differentiable *f (x, y)* with respect to y.

Higher Ordered Derivatives

$$(f_x)_x=f_{xx}=f_{11}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y=f_{xy}=f_{12}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\,\partial x}=\frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x=f_{yx}=f_{21}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 z}{\partial x\partial y}$$

$$(f_y)_y=f_{yy}=f_{22}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 z}{\partial y^2}$$

Equation of Tangent Planes:

An equation to the tangent plane to the surface $z=f(x,y)$ at the point $P(x_0,y_0,z_0)$ is

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Equation of normal line to the surface through:

$$\frac{x-x_0}{\left(\frac{\partial f}{\partial x}\right)_P}=\frac{y-y_0}{\left(\frac{\partial f}{\partial y}\right)_P}=\frac{z-z_0}{\left(\frac{\partial f}{\partial z}\right)_P}$$

Differential (Total derivative):

$$dz = f_x(x,y)dx + f_y(x,y)dy$$
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Chain Rule (CASE 1): Suppose that $z = f(x,y)$ is a differentiable function of *x* and *y*, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Chain Rule (CASE 2): Suppose that $z = f(x,y)$ is a differentiable function of *x* and *y*, where $x = g(s,t)$ and $y = h(s,t)$ are both differentiable functions of *s* and *t*. Then z is a differentiable function of *s* and *t*. Then

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Maximum and Minimum Values:

A function of two variables has a local maximum at *(a, b)* if $f(x,y) \le f(a,b)$ when *(x, y)* is near *(a, b)*. The number *f (a, b)* is called a local maximum value. If $f(x,y) \ge f(a,b)$ when *(x, y)* is near to *(a, b)*, then *f (a, b)* is called a local minimum value.

Second Derivative Test: Suppose the second partial derivatives of *f* are continuous on a disk with center *(a, b)*, and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$

(a) If D>0 and $f_{xx}(a,b) > 0$, $f(a,b)$ is a local minimum.

(b) If D>0 and $f_{xx}(a,b) < 0$, $f(a,b)$ is a local maximum.

(c) If D<0 and $f(a,b)$ is neither maximum nor minimum.

Method of Lagrange Multipliers:

To find the maximum and minimum values of $f(x,y,z)$ subject to the constrain $g(x,y,z)=k$

[Assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x,y,z)=k$]

(a) Find all values of x, y ,z and λ such that $\nabla f(x\ y,z)=\lambda \nabla g(x,y,z)$ and $g(x,y,z)=k$

(b) Evaluate *f* at all the points *(x, y, z)* that result from step (a) .The largest of these values is the maximum value of f; the smallest is the minimum value of *f*.

Euler's Theorem:

If *u* is a function of *x* and *y* of degree *n*, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$.

Cor. 1 If *u* is a homogenous function of *x and y* of degree *n*, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Cor. 2 If Z is a homogenous function of degree *n* in *x* and *y* and Z = *f (u)* then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Cor. 3 If Z is a homogenous function of degree *n* in *x* and *y* and Z = *f(u)* then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)(g'(u)-1) \text{ where } g(u)=n\frac{f(u)}{f'(u)}$$

Derivative of an implicit function:

If *f (x, y) = 0* be an implicit function with *y = g(x)*, then $\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{f_x}{f_y}$

Taylor's Expansion for a function of two variables:

$$f(x+h,y+k)=f(x,y)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f+\ldots$$

Cor.1

$$f(a+h,b+k)=f(a,b)+\left[h f_x(a,b)+k f_y(a,b)\right]+\frac{1}{2!}\left[h^2 f_{xx}+2hk f_{xy}(a,b)+k^2 f_{yy}(a,b)\right]+\ldots$$

Cor.2

$$f(a+h,b+k)=$$
$$f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+$$
$$\frac{1}{2!}\left[(x-a)^2 f_{xx}+2(x-a)(y-b)\,f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]+\ldots$$

<u>**Jacobian:**</u>

1. If $u = u(x, y)$ and $v = v(x, y)$ then the Jacobian of u and v w.r.t x and y is given by $J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$

2. If $u = u(x, y, z)$ and $v = v(x, y, z)$ and $w = w(x, y, z)$, then the Jacobian of u and v and w w.r.t x and y and z is given by $J=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$

<u>Properties of Jacobian:</u>

1. If u and v are functions of x and y and x and y are functions of r and s , then $\frac{\partial(u,v)}{\partial(r,s)}=\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(r,s)}$

2. If $J=\frac{\partial(u,v)}{\partial(x,y)}$ and $J'=\frac{\partial(x,y)}{\partial(u,v)}$ then $JJ'=1.$

<u>**Applications:**</u>

❖ In Chemical Engineering, partial derivatives (as rate of change) are used to find temperature on metal plates. Temperature on the plate varies with position. Therefore, it is considered as function of two variables.

- Using partial derivatives, we can study a function of more than one variable, we can find its maximum, minimum, determine the direction in which it is increasing or decreasing without actually plotting the function.

- By using the optimization of functions (Lagrangian multipliers) in just a few steps you can answer very practical and useful questions such as: “You have square piece of cardboard, with sides 1 meter in length. Using that piece of card board, you can make a box, what are the dimensions of a box containing the maximal volume?” When a function of two variables is involved we can use second derivative test to find its maximum and minimum values (optimal values). We can employ the similar concepts of partial derivatives to find optimal values of the function like cost, strength, amount of material used in a building, profit, loss, etc.

- Partial derivatives occur in partial differential equations which are observed in various fields.
  (1) Maxwell's equations of electromagnetism
  (2) Einstein’s general relativity equation for the curvature of space-time given mass-energy- momentum.
  (3) The equation for heat conduction (Fourier)
  (4) The equation for the gravitational potential of a blob of mass (Newton-Laplace)
  (5) The equations of motion of a fluid (gas or liquid) (Euler-Navier-Stokes)
  (6) The Schrodinger equation of quantum mechanics
  (7) The Dirac equation of quantum mechanics
  (8) The Yang-Mills equation
  (9) The Liouville equation of statistical mechanics

- PDEs are used in simulation of real life models like heat flow equation is used for the analysis of temperature distribution in a body, the wave equation for the motion of a waveforms, the flow equation for the fluid flow and Laplace’s equation for an electrostatic potential.

- The famous partial differential equation is the wave or Harmonic equation in physics. Then there is the Navier-Stokes equation (For example, the space shuttle foam problem during lift-off can be modeled with the Navier-Stokes partial differential equation which involves 7 variables in it).All this equations occur many times in practical applications and requires knowledge of partial derivatives.

Integral Calculus

Proper integral:

An integral which has neither limit infinite and from which the integrand does not approach infinity at any point in the range of integration.

Improper Integral:

An integral is an improper integral if either the interval of integration is not finite (improper integral of type 1) or if the function to integrate is not continuous (not bounded) in the interval of integration (improper integral of type 2).

Example:-1. $\int_0^{\infty} e^{-x}dx$. is an improper integral of type 1 since the upper limit of integration is infinite.

Example:-2. $\int_0^{1} \frac{1}{x}dx$. is an improper integral of type 2 because $\frac{1}{x}$ is not continuous at 0.

Example:-3. $\int_0^{\infty} \frac{1}{x-1}dx$. is an improper integral of types 1 since the upper limit of integration is infinite. It is also an improper integral of type 2 because $\frac{1}{x-1}$ is not continuous at 1 and 1 is in the interval of integration.

Reduction formula:

(1) $\int_0^{\frac{\pi}{2}} \sin^m x\,dx = \frac{m-1}{m}\frac{m-3}{m-2}\frac{m-5}{m-4}.......x$ where $x = \begin{cases} 1 & if\ \ m\, is\, odd \\ \frac{\pi}{2} & if\ \ m\, is\, even \end{cases}$

(2) $\int_0^{\frac{\pi}{2}} \cos^m x\,dx = \frac{m-1}{m}\frac{m-3}{m-2}\frac{m-5}{m-4}.......x$ where $x = \begin{cases} 1 & if\ \ m\, is\, odd \\ \frac{\pi}{2} & if\ \ m\, is\, even \end{cases}$

(3) $\int_0^{\frac{\pi}{2}} \cos^m x \sin^n x\,dx = \frac{(m-1)(m-3).........(n-1)(n-3)(n-5)...........}{(m+n)(m+n-2)(m+n-4).............}x$, $x = \begin{cases} \frac{\pi}{2} & if\ m\,\&\,n\, are\ even \\ 1 & ottherwise \end{cases}$

Applications:

The process of integration has interesting application in geometry, physics, and evaluation of series. These are basic tools of engineering and sciences. Integration is used to rectify the curve, evaluate the area of the regions etc. The application of integration is given below:

- ❖ **Rectification:**

A curve whose length can be found is called a rectifiable curve and the process of finding the length of a curve is called rectification.

(i) Length of the plane curve in Cartesian form: Let $y=f(x)$ be the equation of the plane curve. Let S be the length of the arc of the plane curve $y=f(x)$ included between two points A and B whose abscissa are a and b. Then S can be given as

$$S=\int_{a}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx$$

(ii) Length of the plane curve in Polar form: Let $r=f(\theta)$ be the polar equation of the curve. Then the length of the arc of the curve included between two points whose vectorial angles are $\theta=\alpha$ and $\theta=\beta$ is

$$S=\int_{\theta=\alpha}^{\beta}\sqrt{r^{2}+\left(\frac{dr}{d\theta}\right)^{2}}\,d\theta$$

(iii) Length of the plane curve in parametric form: Let $x=f(t)$ and $y=g(t)$ be the equation of the curve in parametric form, where t is a parameter. then the length of the curve between the points $t=t_1$ and $t=t_2$ is

$$S=\int_{t=t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^{2}+\left(\frac{dy}{dt}\right)^{2}}\,dt$$

❖ **Quadrature:** The process of finding the area of a bounded region of a curve is called quadrature.

(i) **Area of a plane region in Cartesian form:** Let $y=f(x)$ be a function defined on the interval $[a,b]$. Let us assume that $f(x)\geq 0$. Then the area A of the curvilinear Trapezoid bounded by the curve $y=f(x)$, the x- axis and the two ordinates $x=a$ and $x=b$ is given by

$$A=\int_{x=a}^{b} f(x)\,dx$$

The area A bounded by the two curves $y=f_1(x)$, $y=f_2(x)$ and the ordinates $x=a$ and $x=b$ is given by

$$\begin{aligned} A &= \int_{x=a}^{b} f_2(x)\,dx-\int_{x=a}^{b} f_1(x)\,dx \\ &= \int_{x=a}^{b}\left[f_2(x)-f_1(x)\right]dx \end{aligned}\quad \text{, provided } f_2(x)\geq f_1(x)$$

To find the area of a closed curve we have to find out the tangents to the curve parallel to the y-axis. Let $x=a$ and $x=b$ be two tangents. Let an intermediate ordinate meet the curve in two points $P_1(x,y)$ and $P_2(x,y)$ where $y_1>y_2$, then the area A of the closed curve is

$$A=\int_{x=a}^{b}(y_1-y_2)\,dx$$

Where the values of y_1 and y_2 corresponding to any value of x are found by solving the equation of the curve as a quadratic in y.

(ii) **Area bounded by a Polar Curve:** The area bounded by the curve $r=f(\theta)$ between the radii vector $\theta=\alpha$ and $\theta=\beta\,(\alpha\leq\beta)$ is given by

$$A=\frac{1}{2}\int_{\theta=\alpha}^{\beta} r^2\,d\theta=\frac{1}{2}\int_{\theta=\alpha}^{\beta}\left[f(\theta)\right]^2 d\theta$$

(iii) **Area bounded by a Parametric Curve:** If $x=f(t)$ and $y=g(t)$, $a\le t\le b$, are the parametric equations of a curve, then the area bounded by the curve, the x-axis and the ordinates $x=a$ and $x=b$ is given by

$$A=\int_{x=a}^{b} f(x)dx=\int_{x=a}^{b} g(t)\frac{d}{dt}\left[f(t)\right]dt$$

❖ **Volume of the solid of revolution:** A solid of revolution is generated when we revolve a plane region R about a line L. The line L is called the axis of revolution. For example, a plane region R bounded by $y=f(x)$, x-axis, $x=a$ and $x=b$ is rotated about x-axis then we get a solid. Such a solid is called solid of revolution.

(a) The volume of the solid generated by the revolution about the x- axis of the area bounded by the curve $y=f(x)$, the ordinates $x=a$, $x=b$ and the x-axis is given by

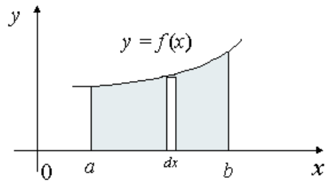
$$V=\pi\int_{a}^{b} y^2 dx\,.$$

(b) The volume of the solid generated by revolution about the y-axis of the area bounded by the curve $x=g(y)$, the abscissa $y=c$, $y=d$ and the x-axis is given by

$$V=\pi\int_{c}^{d} x^2 dy\,.$$

❖ **Centroid for Curved Areas:**

Taking the simple case first, we aim to find the centroid for the area defined by a function $f(x)$, and the 2 vertical lines $x = a$ and $x = b$ as indicated in the following figure.



To find the centroid, we use the same basic idea that we were using for the straight-sided case above. The "typical" rectangle indicated has width Δx and height $y = f(x)$.

Generalizing from the above rectangular areas case, we can find the coordinates $\left(\bar{x}, \bar{y}\right)$ of the centroid using the total moments in the x-direction, given by:

$$\bar{x} = \frac{total\ moments}{total\ area} = \frac{1}{A}\int_a^b xf\left(x\right)dx$$

and, considering the moments in the y-direction about the x-axis and re-expressing the function in terms of y,

$$\bar{y} = \frac{total\ moments}{total\ area} = \frac{1}{A}\int_c^d yf\left(y\right)dy$$

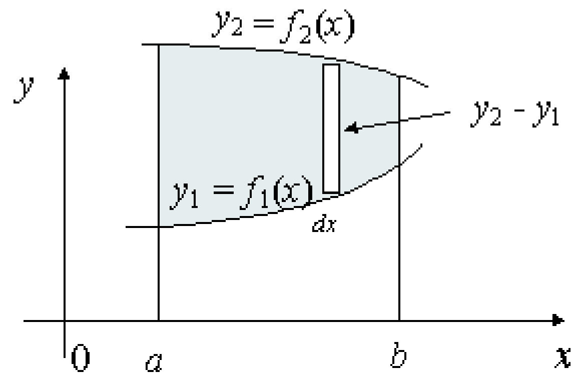
Of course, there may be a rectangular portion to consider separately, as in the given diagram above.

<u>**Alternate method:**</u> Depending on the function, it may be easier to use the following alternative formula for the y-coordinate, which is derived from considering moments in the x-direction (Note the "dx" and the upper and lower limits are along the x-axis):

$$\bar{x} = \frac{total\ moments}{total\ area} = \frac{1}{A}\int_a^b \frac{\left[f\left(x\right)\right]^2}{2}dx$$

Another advantage of this second formula is there is no need to re-express the function in terms of y.

❖ Centroids for Areas Bounded by two Curves:



We extend the simple case given above. The "typical" rectangle indicated has width Δx and height $y_2 - y_1$, so the total moments in the x-direction over the total area is given by:

$$\bar{x} = \frac{total\ moments}{total\ \ area} = \frac{1}{A}\int_a^b x\left(y_2 - y_1\right)dx$$

For the y coordinate, we have 2 different ways we can go about it.

Method 1: We take moments about the y-axis and so we'll need to re-express the expressions x_2 and x_1 as functions of y.

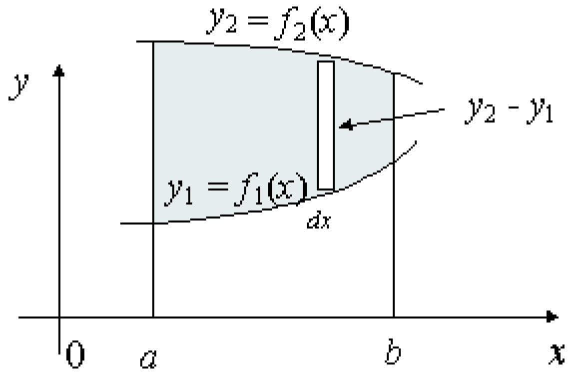
$$\bar{x} = \frac{total\ moments}{total\ \ area} = \frac{1}{A}\int_c^d y\left(x_2 - x_1\right)dy$$

Method 2: We can also keep everything in terms of x by extending the "Alternate Method" given above:

$$\bar{y} = \frac{total\ moments}{total\ \ area} = \frac{1}{A}\int_a^b \frac{\left[y_2\right]^2 - \left[y_1\right]^2}{2}dx$$

❖ Moment of Inertia for Areas:

The moment of inertia, I_y of the given area, which is rotating around the y-axis. Each typical rectangle indicated has width dx and height $y_2 - y_1$, so its area is $\left(y_2 - y_1\right)dx$. if k is the mass per unit area, then each typical rectangle has mass $k\left(y_2 - y_1\right)dx$.



The moment of inertia for each typical rectangle is $\left[k\left(y_2 - y_1\right)dx\right]x^2$, since each rectangle is x units from the y-axis. We can add the moments of inertia for all the typical rectangles making up the area using integration:

$$I_y = k\int_a^b x^2\left(y_2 - y_1\right)dx$$

using a similar process the radius of gyration R_y is given by:

$$R_y = \sqrt{\frac{I_y}{m}} \quad \text{where m is the mass of the area.}$$

- ❖ **Work:** Assume that a constant force F is used to move an object a distance d along a straight line. then we know that the work done by the force is defined to by the product of force and distance. However, this formula does not work when force F is a variable. In such a case we use the method of integration as an efficient tool for calculating work.

Let F be a continuous function on the closed interval [a, b] that represents force. then the work done by the force F in moving an object from $x = a$ and $x = b$ along a straight line is given by

$$W = \int_a^b F\left(x\right)dx, \text{where F is the variable force.}$$

❖ **Average value of a function:**

Currents and voltages often vary with time. Engineers may wish to know the average value of such current or voltage over some particular time interval. The average value of a time-varying function is defined in terms of an integral. An associated quantity is the root mean square of a function. root mean square value of a current is used in the calculation of the power dissipated by a resistor.

Suppose $f(t)$ is a function defined on $a \le t \le b$. The area, *A*, under *f* is given by

$$A = \int_a^b f\, dt \,.$$

A rectangular with base spanning the interval [a, b] and height *h* has an area of $h(b-a)$. Suppose the height, *h*, is chosen so that the area under *f* and the area of the rectangle are equal. This means

$$h(b-a) = \int_a^b f dt$$

$$h = \frac{\int_a^b f dt}{(b-a)}$$

Then h is called the average value of the function across the interval [a, b].

❖ **Root mean square value of a function:**

If $f(t)$ is defined on [a, b], the root mean square value is given by

$$\text{Root mean square} = \sqrt{\frac{\int_a^b \left(f(t)\right)^2 dt}{(b-a)}}$$

Beta-Gamma function:

The common method for determining the value of *n!* is naturally recursive, found by multiplying $1\cdot 2\cdot 3\cdot 4\cdots(n-2)\cdot(n-1)\cdot n$, through this is terribly inefficient for large *n*. Is there an explicit way to determine the value of *n!* which uses elementary algebraic operations? Is it possible to find *n!* for real value numbers? The answer to this question is, "The gamma function". The gamma function was first introduced by Swiss mathematician Leonhard Euler.

Gamma function:

The gamma function (also known as Euler's integral of the second kind) is denoted by $\Gamma(n)$ and defined as

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx;\quad n>0$$

Properties of Gamma function:

(1) $\Gamma(n+1)=n\Gamma(n)\quad ;n>0$.

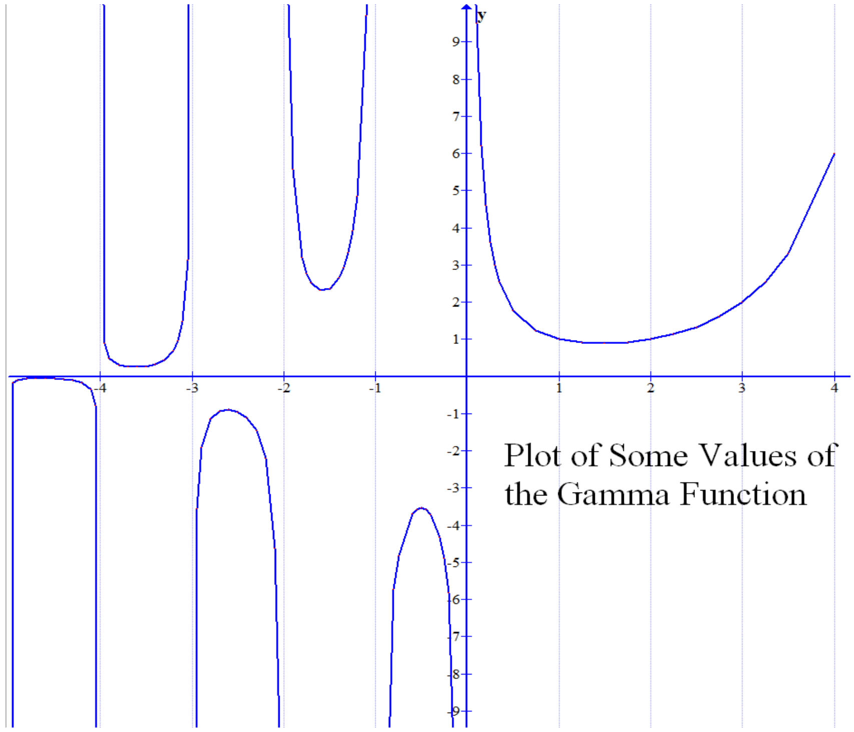
(2) $\Gamma(n+1)=n!$, when n is positive integer.

(3) $\Gamma(n)=2\int_0^\infty e^{-x^2}x^{2n-1}dx\quad ;n>0$.

(4) $\dfrac{\Gamma(n)}{a^n}=\int_0^\infty e^{-ax}x^{n-1}dx\ ;a>0,n>0.$

(5) $\Gamma\left(\dfrac{1}{2}\right)=\sqrt{\pi}$

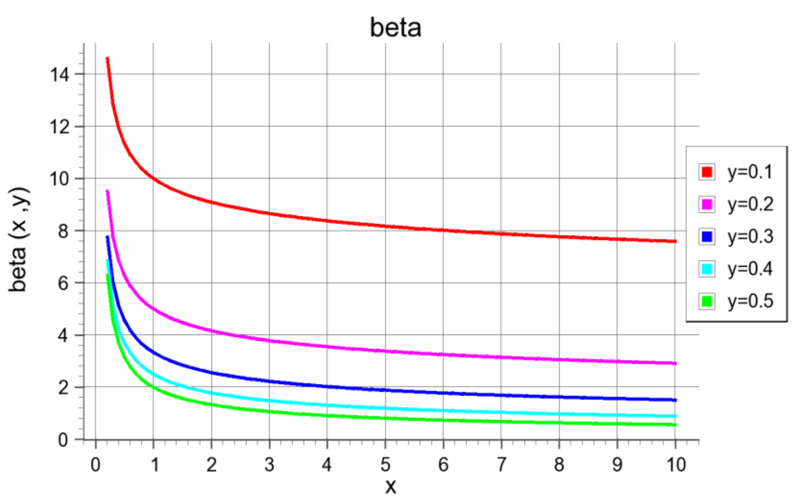
(6) $\int_0^\infty e^{-x^2}dx=\dfrac{\sqrt{\pi}}{2}$



Beta function:

The beta function (also known as Euler's integral of first kind) is denoted by $B(m,n)$ or $\beta(m,n)$ and defined as

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx\ ;m>0,n>0\,.$$



Properties of Beta function:

(1) $B(m,n)=B(n,m)$

(2) $B(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

(3) $B(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

(4) $B(m,n)=\int_0^{1}\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

Relation between Beta and Gamma functions:

$$B(m,n)=\frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Some standard Results:

(1) $\int_0^{\frac{\pi}{2}}\sin^{p}\theta\cos^{q}\theta\,d\theta=\frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.$

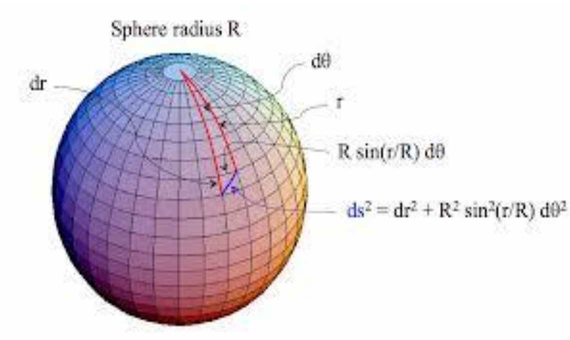
(2) $\Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}\;;0<n<1$ (Euler's formula).

(3) $\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$ (Legendre's formula or Duplication formula)

Application of Beta-Gamma functions:

- The Euler Beta function appeared in elementary particle physics as a model for the Scattering amplitude in the so called "dual resonance model"
- In string theory, the partition function of dense harmonic matter is described in terms of Gamma function.
- In Mechanical engineering, Incomplete Gamma function is used in Transient heat conduction.
- In Mechanical engineering, Gamma function is used in Acceleration field of a fluid.

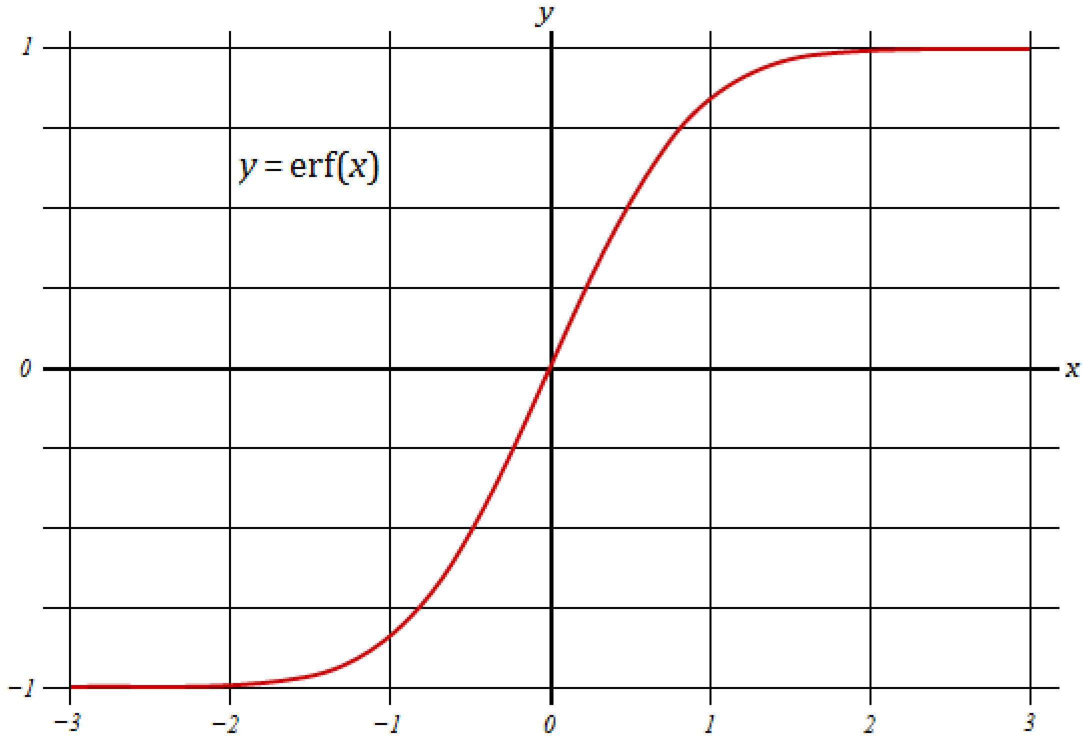
- ❖ Nuclear interactions of elementary particles modeled as one-dimensional strings instead of zero-dimensional particles were described by the Euler beta function.
- ❖ The Gamma function, find application in such diverse area as quantum physics, astrophysics, fluid dynamics, thermodynamics.
- ❖ Beta and Gamma functions are used to evaluate integrals.
- ❖ The solution of Bessel function, hyperbolic Bessel function are involve Gamma function.
- ❖ The gamma function can also be used to calculate "volume" and "area" of n -dimensional hyperspheres.



- ❖ Used in probability density function of Gamma distribution (This Gamma distribution is used in Statistics to model a wide range of process, for example, the time between occurrence of earth quakes),

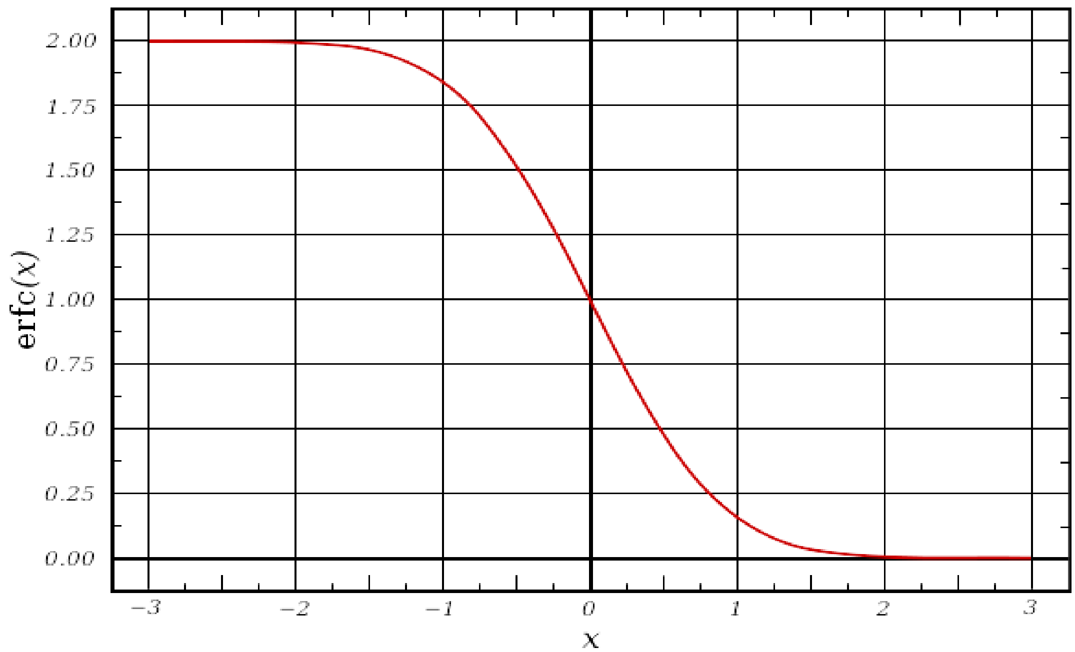
Error function:

The error function of x is defined by $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$



Complimentary error function:

The complimentary error function of x is defined by $erf_c(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$



Properties:

(1) $erf(0)=0$, $erf_c(0)=1$.

(2) $erf(\infty)=1$, $erf_c(\infty)=0$.

(3) $erf(x)+erf_c(x)=1$.

(4) $erf(x)$ is an odd function.

(5) $erf_c(x)+erf_c(-x)=2$.

(6) $erf(x)=\dfrac{2}{\sqrt{\pi}}\left(x-\dfrac{x^3}{1!\,3}+\dfrac{x^5}{2!\,5}-\dfrac{x^7}{3!\,7}+.....\right)$

Applications:

- ❖ In chemical engineering, the error function arises when you are solving for the diffusion of heat through a medium when the heat source is "pulsing".
- ❖ In chemical engineering, the error function is used in working on problems of distribution of velocities of gases in different astrophysical regimes. Molecular velocities are (usually) distributed according to a Gaussian function, and in trying to determine fractions of molecules with speeds in certain ranges, the error function popped up.

- In mechanical engineering, the application of the Error Function is in heat conduction (or in atomic diffusion, mathematically similar phenomena). It's used to describe the transient temperature (or concentration) gradient as a function of distance beneath the surface, *x*, and time, *t*, for a semi-infinite solid. The free surface of this solid is exposed to a different temperature (or atomic concentration) than the bulk. In the analysis the surface (where *x* = 0) temperature never changes. Over time (*t* increasing) the temperature curve inside the sold flattens out and approaches the surface. The shape of this curve is some whacked out integral that they named *erf*.
- In electric engineering, Use of erf or erf_C for solving differential equations include short-circuit power dissipation in electrical engineering.

Multiple integral:

In ordinary integration, we are concerned with the area under the curve $y = f(x)$. Many functions of interest in real life entail several variables, and multiple integrals are a natural extension of the one-dimensional ideas to deal with multivariate problems.

To get a feel for how multiple integrals arise, let's consider a couple of physical example. Suppose that we wish to calculate the force exerted on a wall by a gale. If the pressure P was constant across the whole face with area A, then the total force is simply $P \times A$. With a varying pressure $P(x, y)$, the answer is not obvious. This situation can be handled by thinking about the wall as consisting of many small squares segments, each with area $\delta x \delta y$, so the total force is the sum of all the contributions $P(x, y)\delta x \delta y$; in the limiting case $\delta x \to 0$ and $\delta y \to 0$, we have

$$\text{Force} = \iint_{wall} P(x, y)dxdy,$$

Where the double integral indicates that the infinitesimal summation is being carried out over a two-dimensional surface (in x and y directions).

Incidentally, if the wall does not have a convectional (rectangular) shape then its area can be calculated similarly according to

$$\text{Area} = \iint_{wall} dxdy$$

Another illustration is provided by quantum mechanics where the modulus squared of the wave function, $|\psi(x, y, z)|^2$, of an electron (say) gives the probability density of finding it at some point in space. The chances that the electron is in a small (cuboids) region of volume $\delta x \delta y \delta z$ is then $|\psi(x,y,z)|^2 \delta x \delta y \delta z$. Hence, the probability of finding it within a finite domain V is given by

$$\text{Probability} = \iiint |\psi(x,y,z)|^2 \, dxdydz \,,$$

which is known as Triple integral or volume integral.

❖ Area by double integration:

a) Cartesian coordinates:

The area A of a region R in XOY – plane bounded by the curves $y = f_1(x), y = f_2(x)$ and the lines $x = a, x = b$ is

$$A = \iint_R dy\,dx = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$$

b) Polar coordinates

The area A of a region R bounded by the curves $r = f_1(\theta), r = f_2(\theta)$ and the radii vectors $\theta = \alpha, \theta = \beta$ is

$$A = \iint_R dx\,dy = \int_\alpha^\beta r\,dr\,d\theta$$

❖ Volume as a double integrals:

$V = \iint z\,dx\,dy$, Where $z = f(x,y)$ in Cartesian form

$V = \iint z\,r\,dr\,d\theta$, Where $z = f(r,\theta)$ in polar form

❖ Volume of solid of revolution:

$V = 2\pi \iint y\,dy\,dx$, Where $y = f(x)$ is a plane curve in Cartesian form.

$V = 2\pi \iint r^2 \sin\theta\,d\theta\,dr$, Where $r = f(\theta)$ is polar curve (when rotated about initial line)

$V = 2\pi \iint r^2 \cos\theta \, d\theta \, dr$, Where $r = f(\theta)$ is polar curve (when rotated about the line $\theta = \frac{\pi}{2}$)

<u>Volume as a triple integral:</u>

The volume V of a three dimensional region is $V = \iiint dx\,dy\,dz$

❖ **<u>Centre of Gravity:</u>**

a) <u>Centre of gravity (centroid) of a lamina</u>

If ρ be the surface density, $dm = \rho \, dA$ and if the curve is given in Cartesian coordinates then the C.G. of lamina are

$$\bar{x} = \frac{\iint x \rho \, dx \, dy}{\iint \rho \, dx \, dy}, \qquad \bar{y} = \frac{\iint y \rho \, dx \, dy}{\iint \rho \, dx \, dy}$$

If the curve is given in polar coordinates $x = r\cos\theta, y = r\sin\theta$, $dA = r\,dr\,d\theta$ then

$$\bar{x} = \frac{\iint r^2 \cos\theta \, dr \, d\theta}{\iint r \, dr \, d\theta}, \qquad \bar{y} = \frac{\iint r^2 \sin\theta \, dr \, d\theta}{\iint r \, dr \, d\theta}$$

b) <u>Centre of gravity of solid:</u>

If ρ be the volume density, $dm = \rho \, dx \, dy \, dz$ then the coordinates of the C.G. of solid are

$$\bar{x} = \frac{\iint x \rho \, dx \, dy \, dz}{\iint \rho \, dx \, dy \, dz}, \quad \bar{y} = \frac{\iint y \rho \, dx \, dy \, dz}{\iint \rho \, dx \, dy \, dz}, \quad \bar{z} = \frac{\iint z \rho \, dx \, dy \, dz}{\iint \rho \, dx \, dy \, dz}$$

❖ **<u>Moment of Inertia</u>**

a) <u>Moment of inertia of plane lamina</u>

Let A be the area of a plane lamina and ρ its density, then the moment of inertia of an area A about the $X-$ axis is

$$I_x = \iint_A \rho\, y^2\, dA = \iint_A \rho\, y^2\, dx\, dy$$

The moment of inertia of an area A about the $Y-$axis is

$$I_y = \iint_A \rho\, x^2\, dA = \iint_A \rho\, x^2\, dx\, dy$$

The moment of inertia of an area A about an axis through the origin and perpendicular to the $XY-$plane is

$$I_o = I_x + I_y = \iint_A \rho \left(x^2 + y^2\right) dx\, dy$$

b) <u>Moment of inertia of a solid</u>

Let V be the volume of a solid and ρ its density, then the moment of inertia of a solid about the $X-$ axis is

$$I_x = \iiint_V \rho \left(y^2 + z^2\right) dx\, dy\, dz$$

The moment of inertia of a solid about $Y-$axis is

$$I_y = \iiint_V \rho \left(z^2 + x^2\right) dx\, dy\, dz$$

The moment of inertia of a solid about $Z-$axis is

$$I_z = \iiint_V \rho \left(x^2 + y^2\right) dx\, dy\, dz$$

❖ **<u>Mean values:</u>**

The mean value of the function $z = f(x, y)$ over a region of an area A is given by

M.V. of z over an area A = $z_m = \dfrac{\iint_R f(x, y)\, dx\, dy}{\iint_R dx\, dy}$

The mean value of the function $u = f(x, y, z)$ over a region of volume V is given by M.V. of z over an volume V = $u_m = \dfrac{\iiint\limits_V f(x,y,z)\,dx\,dy\,dz}{\iiint\limits_V dx\,dy\,dz}$

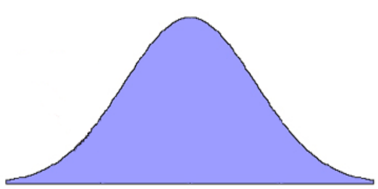
<u>Applications</u>:

- In the part of vector integration, multiple integrals are useful in defining line integrals, surface integrals & volume integrals, and as a part of its application multiple integrals are useful in Gauss Divergence theorem, Stoke's theorem and Green's theorem.
- Multiple integrals are useful in probability to define joint density function. For example, consider a pair of continuous random variables X and Y, such as the lifetimes of two components of a machine or the height and weight of an adult female chosen at random. The joint density function of X and Y is a function f of two variables such that the probability that (X,Y) lies in region D is:

$$P\left((X,Y)\in D\right)=\iint\limits_D f(x,y)\,dA$$

- Finding the Average of a Function using double integrals: A function can represent many things. One example is the path of an airplane. Using calculus (Multiple integrals) you can calculate its average cruising altitude, velocity and acceleration. Same goes for a car, bus, or anything else that moves along a path.

- <u>Calculating the Area of Any Shape:</u> Although we do have standard methods to calculate the area of some shapes, calculus allows us to do much more. Trying to find the area on a shape like this would be very difficult if it wasn't for calculus.



- Green's Theorem, which gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C, is applied in an instrument known as a planimeter, which is used to calculate the area of a flat surface on a drawing. For example, it can be used to calculate the amount of area taken up by an irregularly shaped flower bed or swimming pool when designing the layout of a piece of property.
- Discrete Green's Theorem, which gives the relationship between a double integral of a function around a simple closed rectangular curve *C* and a linear combination of the anti derivative's values at corner points along the edge of the curve, allows fast calculation of sums of values in rectangular domains. For example, it can be used to efficiently calculate sums of rectangular domains in images, in order to rapidly extract features and detect object.
- Differential form and integral form of physical laws: As a result of the divergence theorem, a host of physical laws can be written in both a differential form (where one quantity is the divergence of another) and an integral form (where the flux of one quantity through a closed surface is equal to another quantity). Three examples are Gauss's law (in electrostatics), Gauss's law for magnetism, and Gauss's law for gravity.
- Continuity equations: Continuity equations offer more examples of laws with both differential and integral forms, related to each other by the divergence theorem. In fluid dynamics, electromagnetism, quantum mechanics, relativity theory, and a number of other fields, there are continuity equations that describe the conservation of mass, momentum, energy, probability, or other quantities. Generically, these equations state that the divergence of the flow of the conserved quantity is equal to the distribution of *sources* or *sinks* of that quantity. The divergence theorem states that any such continuity equation can be written in a differential form (in terms of a divergence) and an integral form (in terms of a flux).
- Inverse-square laws: Any inverse-square law can instead be written in a Gauss' law-type form (with a differential and integral form, as described above). Two examples are Gauss' law (in electrostatics), which follows from the inverse-square Coulomb's law, and Gauss' law for gravity, which follows from the inverse-square Newton's law of universal gravitation. The derivation of the Gauss' law-type equation from the inverse-square formulation (or vice-versa) is exactly the same in both cases.
- The main thing in electromagnetic is Maxwell equation which is based on this stokes theorem and divergence theorem. When we calculate stoke theorem then definitely double integrals is used.(For more information visit the following site:http://www.scribd.com/doc/97929011/Multiple-Integrals-and-Its-Application-in-Telecomm-Engineering)